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Extended Hill Equation

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If $J(z)$ is a periodic even function of z with period π, the equation $d^2u(z)/dz^2 + J(z)u(z) = 0$ is Hill's equation. The solution is obtained to an extended Hill equation where $J(z)$ is a periodic real function of z, using Floquet's method.

While Hill's equation is important in mathematical physics, many physical problems do not satisfy its conditions. The purpose of this paper is to extend Floquet's method to solve an extended Hill equation.

Considering the following equation:

$$\frac{d^2u}{dz^2} + J(z)u = 0 \tag{1}$$

If $J(z)$ is an even function of z with period π, this is Hill's equation, and the solutions have been studied in Floquet's theory (Wang and Guo, 1965; Whittaker and Watson, 1927; McLachlan, 1951). Here the solution when $J(z)$ is a periodic real function of z is obtained following Floquet's method.

As a start, one may study the case where $J(z)$ is a periodic real function of z with period π. When z is real, $J(z)$ can be expanded in the form

$$J(z) = \sum_{n=-\infty}^{\infty} a_{2n} \exp(i2nz)$$
$$a_{2n} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dz\, J(z) \exp(-2inz) \qquad (n = 0, \pm 1, \pm 2, \ldots) \tag{2}$$

That $J(z)$ is a real function of z yields

$$a_{-2n} = a_{2n}^* \qquad (n = 1, 2, 3, \ldots) \tag{3}$$

We assume that the solution can be written in the form

$$u(z) = \exp(i\mu z) \sum_{n=-\infty}^{\infty} b_n \exp(inz) = \sum_{n=-\infty}^{\infty} b_n \exp(i\mu z + inz) \tag{4}$$

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Substituting equations (2) and (4) into equation (1), one obtains

$$-\sum_{n=-\infty}^{\infty} b_n(\mu+n)^2 \exp(i\mu z+inz) + \sum_{k=-\infty}^{\infty} a_{2k} \exp(i2kz) \sum_{n=-\infty}^{\infty} b_n \exp(i\mu z+inz)=0 \tag{5}$$

Thus, the correlation equation system for the coefficients b_n is

$$-(\mu+n)^2 b_n + \sum_{k=-\infty}^{\infty} a_{2k} b_{n-2k}=0 \qquad (n=0, \pm 1, \pm 2, \ldots) \tag{6}$$

The cases $n=$ even and $n=$ odd will be discussed separately.

(I) For $n=$ even, let $n=2l$; by dividing the linear equations of (6) by $a_0-(\mu+2l)^2$, one can obtain the equation system after the variable transform $m=l-k$:

$$\sum_{m=-\infty}^{\infty} B_{lm} b_{2m}=0 \qquad (l=0, \pm 1, \pm 2, \ldots) \tag{7}$$

where B_{lm} are coefficients defined by

$$B_{lm}=\begin{cases} 1, & l=m \\ \dfrac{a_{2(l-m)}}{a_0-(\mu+2l)^2}, & l \neq m \end{cases} \qquad (l, m=0, \pm 1, \pm 2, \ldots) \tag{8}$$

The nonzero solution to equations (7) yields $\det|B_{lm}|=0$, i.e.,

$$\Delta_1(\mu)=\begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \dfrac{a_2^*}{a_0-(\mu-4)^2} & \dfrac{a_4^*}{a_0-(\mu-4)^2} & \dfrac{a_6^*}{a_0-(\mu-4)^2} & \dfrac{a_8^*}{a_0-(\mu-4)^2} & \cdots \\ \cdots & \dfrac{a_2}{a_0-(\mu-2)^2} & 1 & \dfrac{a_2^*}{a_0-(\mu-2)^2} & \dfrac{a_4^*}{a_0-(\mu-2)^2} & \dfrac{a_6^*}{a_0-(\mu-2)^2} & \cdots \\ \cdots & \dfrac{a_4}{a_0-\mu^2} & \dfrac{a_2}{a_0-\mu^2} & 1 & \dfrac{a_2^*}{a_0-\mu^2} & \dfrac{a_4^*}{a_0-\mu^2} & \cdots \\ \cdots & \dfrac{a_6}{a_0-(\mu+2)^2} & \dfrac{a_4}{a_0-(\mu+2)^2} & \dfrac{a_2}{a_0-(\mu+2)^2} & 1 & \dfrac{a_2^*}{a_0-(\mu+2)^2} & \cdots \\ \cdots & \dfrac{a_8}{a_0-(\mu+4)^2} & \dfrac{a_6}{a_0-(\mu+4)^2} & \dfrac{a_4}{a_0-(\mu+4)^2} & \dfrac{a_2}{a_0-(\mu+4)^2} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}=0 \tag{9}$$

It is easy to see (i) that $\Delta_1(\mu)$ is an even periodic function of μ with period 2, and (ii) that $\Delta_1(\mu)$ is an analytic function of μ (except at its obvious simple poles), which tends to unity as the imaginary part of μ tends to $\pm\infty$.

If now one chooses the constant K_1 so that the function $D_1(\mu)$, defined by the equation

$$D_1(\mu) \equiv \Delta_1(\mu) - K_1\left[\cot\frac{\pi}{2}(\mu - \sqrt{a_0}) - \cot\frac{\pi}{2}(\mu + \sqrt{a_0})\right] \tag{10}$$

has no pole at the point $\mu = \sqrt{a_0}$, then, since $D_1(\mu)$ is an even periodic function of μ, it follows that $D_1(\mu)$ has no pole at any of the points $2n \pm \sqrt{a_0}$, where n is any integer.

The function $D_1(\mu)$ is therefore a periodic function of μ (with period 2) which has no poles, and which is obviously bounded as $\text{Im}(\mu) \to \pm\infty$. The conditions postulated in Liouville's theorem are satisfied, and so $D_1(\mu)$ is a constant; making $\mu \to +\infty$, one can see that this constant is unity. Therefore, putting $\mu = 0$, one determines

$$K_1 = \frac{1 - \Delta(0)}{2\cot[(\pi/2)\sqrt{a_0}]} \tag{11}$$

On substitution in equation (10), after simple deductions, one obtains the determinantal equation

$$\sin^2\left(\frac{\mu\pi}{2}\right) = \Delta_1(0)\sin^2\left(\frac{\sqrt{a_0}\,\pi}{2}\right) \tag{12}$$

which is similar to Hill's determinantal equation. The roots of equation (12) are the exponents of solution (4).

(II) For $n = \text{odd}$, let $n = 2l + 1$; by dividing the linear equations of (6) by $a_0 - (\mu + 2l + 1)^2$, one obtains the equation system

$$\sum_{m=-\infty}^{\infty} C_{lm} b_{2m+1} = 0 \qquad (l = 0, \pm 1, \pm 2, \ldots) \tag{13}$$

where C_{lm} are coefficients defined by

$$C_{lm} = \begin{cases} 1, & l = m \\ \dfrac{a_{2(l-m)}}{a_0 - (\mu + 2l + 1)^2}, & l \neq m \end{cases} \qquad (l, m = 0, \pm 1, \pm 2, \ldots) \tag{14}$$

One also obtains $\det|C_{lm}| = 0$, i.e.,

$$\Delta_2(\mu) = \begin{vmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & \dfrac{a_2^*}{a_0 - (\mu - 3)^2} & \dfrac{a_4^*}{a_0 - (\mu - 3)^2} & \dfrac{a_6^*}{a_0 - (\mu - 3)^2} & \cdots \\ \cdots & \dfrac{a_2}{a_0 - (\mu - 1)^2} & 1 & \dfrac{a_2^*}{a_0 - (\mu - 1)^2} & \dfrac{a_4^*}{a_0 - (\mu - 1)^2} & \cdots \\ \cdots & \dfrac{a_4}{a_0 - (\mu + 1)^2} & \dfrac{a_2}{a_0 - (\mu + 1)^2} & 1 & \dfrac{a_2^*}{a_0 - (\mu + 1)^2} & \cdots \\ \cdots & \dfrac{a_6}{a_0 - (\mu + 3)^2} & \dfrac{a_4}{a_0 - (\mu + 3)^2} & \dfrac{a_2}{a_0 - (\mu + 3)^2} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \tag{15}$$

Similarly, by introducing a new function

$$D_2(\mu) \equiv \Delta_2(\mu) - K_2\left[\tan\frac{\pi}{2}(\mu - \sqrt{a_0}) - \tan\frac{\pi}{2}(\mu + \sqrt{a_0})\right] \tag{16}$$

one obtains another determinantal equation:

$$\sin^2\left(\frac{\mu\pi}{2}\right) = \Delta_2(0)\cos^2\left(\frac{\sqrt{a_0}\,\pi}{2}\right) \tag{17}$$

One can prove that equations (12) and (17) are not independent. Directly calculating the value of $\Delta_1(0)/\Delta_2(0)$, it is easy to see

$$\frac{\Delta_1(0)}{\Delta_2(0)} = \left[\frac{1}{\sqrt{a_0}}\prod_{n=1}^{\infty}\frac{a_0-(2n-1)^2}{a_0-(2n)^2}\right]^2 = \cot^2\left(\frac{\sqrt{a_0}\,\pi}{2}\right) \tag{18}$$

so that equation (17) is equivalent to equation (12).

All the approximation methods for calculating Hill's determinantal equation can be employed to solve equation (12).

All other coefficients b_n $(n = -1, \pm 2, \pm 3, \ldots)$ (except b_0 and b_1) are determined by b_0 or b_1 through relations (6).

For the more general case where $J(z)$ is a periodic real function of z, a simple variable transformation can be carried out to change the period to be π. Similar results are obtained.

In summary, the solution to an extended Hill equation has been obtained following Floquet's method.

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